UDC 519.213.2+517.53
DOI https://doi.org/10.32838/2663-5941/2021.3/21

Mulyava O.M.
National University of Food Technologies

Sheremeta M.M.
Ivan Franko National University of Lviv

COMPOSITION OF PROBABILITY LAWS

A non-negative and non-decreasing function F continuous on the left on $(-\infty,+\infty)$ is said to be a probability law if $\lim\limits_{x\to+\infty} F(x)=1$ and $\lim\limits_{x\to-\infty} F(x)=0$ and a composition of two probability laws F_1 and F_2 is defined by the equality $F(s)=(F_1*F_2)(s):=\int_{-\infty}^{\infty} F_1(x-s)dF_2(x)$. If for $x\geq 0$ we put $W_F(x)=1-F(x)+F(-x)$ then $W_F(x)\downarrow 0$ as $x\to+\infty$. The article studies the relationship between a decreasing of the function $W_{F_1*F_2}$. and a decreasing of the functions $W_{F_1}(x)$ and $W_{F_2}(x)$ in terms of generalized orders and convergence classes. For this purpose, by L we denote a class of continuous nonnegative on $(-\infty, +\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\leq x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each fixed $c\in(0,+\infty)$, i.e. α is slowly increasing function. Putting $R_F=\varliminf\limits_{x\to+\infty}((1/x)\ln(1/W_f(x))$, two cases $R_F=+\infty$ and $R_F<+\infty$ are considered separately.

For $R_F=+\infty$ the following characteristic $\omega_{\alpha,\beta}[F]:=\varlimsup\limits_{x\to+\infty}\alpha(x)/\beta((1/x)\cdot\ln(1/W_F(x)))$ is introduced and it is proved that if $\alpha\in L_{si}$ and $\beta\in L^0$ then $\omega_{\alpha,\beta}[F_1*F_2]\leq\max\{\omega_{\alpha,\beta}[F_1],\omega_{\alpha,\beta}[F_2]\}$ and, moreover, if $\omega_{\alpha,\beta}[F_2]<\omega_{\alpha,\beta}[F_1]$ then $\omega_{\alpha,\beta}[F_1*F_2]=\omega_{\alpha,\beta}[F_1]$. If $0<R_F=R<+\infty$ and $\varlimsup\limits_{x\to+\infty}W_F(x)e^{Rx}=+\infty$ we put $\omega^{(R)}_{\alpha,\beta}[F]=\varlimsup\limits_{x\to+\infty}\alpha(x)/\beta(x/\ln^+(W_F(x)\cdot e^{R\cdot x}))$. It is proved that if $R_{F_1}=R_{F_2}=R\in(0,+\infty)$, $\alpha\in L_{si}$, $\beta\in L_{si}$, $\alpha(x\alpha^{-1}(c\beta(x))=(1+o(1))c\beta(x)$ and $\alpha(x/\beta^{-1}(c\alpha(x))=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0;+\infty)$ then $\omega^{(R)}_{\alpha,\beta}[F_1*F_2]\leq\max\{\omega^{(R)}_{\alpha,\beta}[F_1],\omega^{(R)}_{\alpha,\beta}[F_2]\}$ and, moreover, if $\omega^{(R)}_{\alpha,\beta}[F_2]<\omega^{(R)}_{\alpha,\beta}[F_1]$ then $\omega^{(R)}_{\alpha,\beta}[F_1*F_2]=\omega^{(R)}_{\alpha,\beta}[F_1]$.

The connection between the decrease of the function $W_{F_1*F_1}(x)$ and the decrease of the functions $W_{F_1}(x)$ and $W_{F_2}(x)$ also is studied in terms of classes of convergence. Under some conditions on the functions α, β and $W_{F_j}(x)$ it is proved, for example, that if $R_F=+\infty$ and $\int_{x_0}^{\infty}\alpha'(x)\beta_1((1/x)\cdot\ln(1/W_{F_j}(x)))dx<+\infty$ for $j=1;2$, where $\beta_1(x)=\int_x^{\infty}dr/\beta(r)$, then $\int_{x_0}^{\infty}\alpha'(x)\beta_1((1/x)\cdot\ln(1/W_{F_1*F_2}(x)))dx<+\infty$.

Key words: *probability law, composition of probability laws, generalized order, convergence class, decrease of function.*

Formulation of the problem. For $x\geq 0$ and probability laws F_j let $W_{F_j}(x)=1-F_j(x)+F_j(-x)$ $(j=1;2)$ In terms of generalized orders and convergence classes connections between the decrease of $W_{F_j}(x)$ and $W(x)$ are established, where $F(s)=(F_1*F_2)(s):=\int_{-\infty}^{\infty}F_1(x-s)dF_2(x)$.

Analysis of recent research and publications. A non-decreasing function F continuous on the left on $(-\infty,\infty)$ is said [5, p. 10] to be a probability law if $\lim\limits_{x\to+\infty}F(x)=1$ and $\lim\limits_{x\to-\infty}F(x)=0$.

If for $x\geq 0$ we put $W_F(x)=1-F(x)+F(-x)$ then $W_F(x)\downarrow 0$ as $x\to+\infty$. A composition of two probability laws F_1 and F_2 is defined [5, p. 10] by the equality $F(s)=(F_1*F_2)(s):=\int_{-\infty}^{\infty}F_1(x-s)dF_2(x)$.

Formulation of the problem. The aim of our note is research of connections between the decrease of function $W_{F_1*F_2}$ and the decrease of functions W_{F_1}

and W_{F_2} in terms of generalized orders and convergence classes.

Statement of basic materials. 1. Connections in terms of generalized orders. We put $R_F = \varliminf_{x\to+\infty} \frac{1}{x}\ln\frac{1}{W_F(x)}$ and we will distinguish between two cases $R_F = +\infty$ and $R_F < +\infty$, and for the research of the decrease of the function W_F we will use generalized orders. With this purpose we denote by L a class of positive continuous functions α on $(-\infty,\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $-\infty < x \le x_0$ and $\alpha(x)\uparrow +\infty$ as $x_0 < x \to +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x\to+\infty$; further, $\alpha \in L_{si}$, if $\alpha\in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x\to+\infty$ for any $c\in(0,+\infty)$, i. e. α is slowly increasing. It easy to see $L_{si}\subset L^0$.

We start from the case $R_F = +\infty$. For $\alpha\in L$, $\beta\in L$ and probability law we define

$$\omega_{\alpha,\beta}[F] := \varlimsup_{x\to+\infty}\frac{\alpha(x)}{\beta\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)}.$$

Theorem 1. If $\alpha\in L_{si}$ and $\beta\in L^0$ then $\omega_{\alpha,\beta}[F_1 * F_2] \le \max\{\omega_{\alpha,\beta}[F_1], \omega_{\alpha,\beta}[F_2]\}$, and moreover if $\omega_{\alpha,\beta}[F_2] < \omega_{\alpha,\beta}[F_1]$ then $\omega_{\alpha,\beta}[F_1 * F_2] = \omega_{\alpha,\beta}[F_1]$.

Proof. Let $\varphi(z) = \int_{-\infty}^{+\infty} e^{izx}dF(x)$ be the characteristic function of probability law F defined [5, p. 12] on real z. If ϕ has an analytic continuation on the disk $D_R = \{z : |z| < R\}$, $0 < R \le +\infty$, then we call φ an analytic in D_R characteristic function of the law F. Further we always assume that D_R is the maximal disk of the analicity of φ. It is known [5, p. 37-38] that ϕ is an analytic in D_R characteristic function of the law F if and only if $W_F(x) = O(e^{-rx})$ as $0 \le x \to +\infty$ for every $r\in[0;R)$. Hence $\varliminf_{x\to+\infty}\frac{1}{x}\ln\frac{1}{W_F(x)} = R$, i. e. $R = R_F$ and if $R_F = +\infty$ then ϕ is an entire function.

Let $M(r,\phi) = \max\{|\varphi(x)| : \ |z| = r\}$ and $\rho_{\alpha,\beta}[\varphi] := \varlimsup_{r\to+\infty}\frac{1}{\beta(r)}\alpha\left(\frac{\ln M(r,\varphi)}{r}\right)$, $\alpha\in L, \beta\in L$ be a generalized order of the function φ. In [3] is proved that if either $\alpha\in L_{si}$ and $\beta\in L^0$ or $\alpha\in L^0$ and $\beta\in L_{si}$ then $\rho_{\alpha,\beta}[\varphi] = \omega_{\alpha,\beta}[F]$.

On the other hand [5, p. 13], if $F = F_1 * F_2$ then for the corresponding characteristic functions the equality $\varphi(z) = \varphi_1(z)\cdot\varphi_2(z)$ is true.

Therefore, we need to prove that

$$\rho_{\alpha,\beta}[\varphi] \le \max\{\rho_{\alpha,\beta}[\varphi_1], \rho_{\alpha,\beta}[\varphi_2]\}, \tag{1}$$

and if $\rho_{\alpha,\beta}[\varphi_2] < \rho_{\alpha,\beta}[\varphi_1]$ then

$$\rho_{\alpha,\beta}[\varphi] = \rho_{\alpha,\beta}[\varphi_1] \tag{2}$$

At first we suppose that $\max\{\rho_{\alpha,\beta}[\varphi_1], \rho_{\alpha,\beta}[\varphi_2]\} = \rho < +\infty$. Then for every $\varepsilon > 0$ and all $r \ge r_0(\varepsilon)$

$$\frac{\ln M(r,\varphi_j)}{r} \le \alpha^{-1}\left((\rho_{\alpha,\beta}[\varphi_j]+\varepsilon)\beta(r)\right) \le \le \alpha^{-1}\left((\rho+\varepsilon)\beta(r)\right),$$

and in view of the equality $\varphi(z) = \varphi_1(z)\cdot\varphi_2(z)$ we have

$$\frac{\ln M(r,\varphi)}{r} \le \frac{\ln M(r,\varphi_1)}{r} + \frac{\ln M(r,\varphi_2)}{r} \le 2\alpha^{-1}\left((\rho+\varepsilon)\beta(r)\right), \quad j = 1; 2.$$

Since $\alpha\in L_{si}$, hence it follows that $\rho_{\alpha,\beta}[\varphi] \le \rho+\varepsilon$, and in view of the arbitrariness of ε we obtain the inequality $\rho_{\alpha,\beta}[\varphi] \le \rho$, which is obvious when $\rho = +\infty$. Inequality (1) is proved.

If $\rho_{\alpha,\beta}[\varphi_2] < \rho_{\alpha,\beta}[\varphi_1]$ then (1) implies the inequality $\rho_{\alpha,\beta}[\varphi] < \rho_{\alpha,\beta}[\varphi_1]$. In order to prove a contrary inequality we write down $\varphi_1(z) = \varphi(z)/\varphi_2(z)$ and use results of value distribution theory.

Let $T(r,f)$ be Nevanlinna characteristic of the function f meromorphic in the disk D_R, $0 < R \le +\infty$. It is know [2, p. 45] that if f_1 and f_2 are meromorphic functions in D_R and $f(z) = f_1(z)\cdot f_2(z)$ then $T(r,f) \le T(r,f_1) + T(r,f_2)$ and $T(r,1/f) = T(r,f) + o(1)$ as $r\uparrow R$.

Therefore,

$$T(r,\varphi_1) \le T(r,\varphi) + T(r,1/\varphi_2) = = T(r,\varphi) + T(r,\varphi_2) + o(1), \quad r\uparrow R_F \tag{3}$$

On the other hand, if the function f is analytic in D_R then [2, p. 54] for $0 < r_1 < r_2 < R$

$$T(r_1,f) \le \ln^+ M(r_1,f) \le \frac{r_2+r_1}{r_2-r_1}T(r_2,f). \tag{4}$$

Since $R = R_F = +\infty$, choosing $r_1 = r$ and $r_2 = (1+\delta)r$, $\delta > 0$, from (4) for the function φ_1 we obtain $T(r,\varphi_1) \le \ln^+ M(r,\varphi_1) \le \frac{2+\delta}{\delta}T((1+\delta)r,\varphi_1)$, whence in view of (3)

$$\frac{\delta}{(2+\delta)(1+\delta)}\frac{\ln M\left(r/(1+\delta),\varphi_1\right)}{r/(1+\delta)}\le$$

$$\frac{T(r,\varphi_1)}{r}\le\frac{\ln M(r,\varphi)}{r}+\frac{\ln M(r,\varphi_2)}{r}+o(1),$$

$r\to+\infty$ and in view of the conditions $\alpha\epsilon L_{si}$ we get

$$\overline{\lim_{r\to+\infty}}\frac{1}{\beta(r)}\alpha\left(\frac{\ln M\left(r/(1+\delta),\varphi_1\right)}{r/(1+\delta)}\right)\le\max\{\rho_{\alpha,\beta}[\varphi],\rho_{\alpha,\beta}[\varphi_2]\},$$

On the other hand,

$$\overline{\lim_{r\to+\infty}}\frac{1}{\beta(r)}\alpha\left(\frac{\ln M\left(r/(1+\delta),\varphi_1\right)}{r/(1+\delta)}\right)=$$

$$=\overline{\lim_{r\to+\infty}}\frac{1}{\beta\left(r/(1+\delta)\right)}\alpha\left(\frac{\ln M\left(r/(1+\delta),\varphi_1\right)}{r/(1+\delta)}\right)\ge$$

$$\ge\overline{\lim_{r\to+\infty}}\frac{1}{\beta(r)}\alpha\left(\frac{\ln M(r,\varphi_1)}{r}\right)\underline{\lim_{r\to+\infty}}\frac{\beta(r)}{\beta((1+\delta)r)}=$$

$$=\rho_{\alpha,\beta}[\varphi_1]\underline{\lim_{r\to+\infty}}\frac{\beta(r)}{\beta((1+\delta)r)}.$$

Thus

$$\rho_{\alpha,\beta}[\varphi_1]\le\max\{\rho_{\alpha,\beta}[\varphi],\rho_{\alpha,\beta}[\varphi_2]\}\overline{\lim_{r\to+\infty}}\frac{\beta((1+\delta)r)}{\beta(r)}.$$

Since $\beta\in L^0$, we have [10] $B[\delta]=\overline{\lim_{r\to+\infty}}\beta((1+\delta)r)/\beta(r)\downarrow 1$ as $\delta\downarrow 0$. Therefore, in view of the arbitrariness of δ we obtain the inequality $\rho_{\alpha,\beta}[\varphi_1]\le\max\{\rho_{\alpha,\beta}[\varphi],\rho_{\alpha,\beta}[\varphi_2]\}$, and since $\rho_{\alpha,\beta}[\varphi_1]>\rho_{\alpha,\beta}[\varphi_2]$, we get $\rho_{\alpha,\beta}[\varphi_1]\le\rho_{\alpha,\beta}[\varphi]$, i. e. (2) holds. Theorem 1is proved.

We remark that for example the functions $\alpha(x)\equiv\ln x$ and $\beta(x)\equiv x$ for $x\ge x_0$ satisfy the conditions of Theorem 1.

Now we consider the case $0<R_F=R<\infty$. Suppose that

$$\overline{\lim_{x\uparrow R_F}}W_F(x)e^{Rx}=+\infty,\qquad(5)$$

and for the study of the asymptotic behavior of $W_F(x)e^{Rx}$ we put

$$\omega_{\alpha,\beta}^{(R)}[F]:=\overline{\lim_{x\to+\infty}}\frac{\alpha(x)}{\beta\left(x/\ln^+\left(W_F(x)e^{Rx}\right)\right)}.$$

As in [2], the generalized order of an analytic in D_R, $0<R<+\infty$, characteristic function φ of probability law F we define by the formula

$$\rho_{\alpha,\beta}^{(R)}[\varphi]:=\overline{\lim_{r\uparrow R}}\frac{\alpha(\ln M(r,\varphi))}{\beta(1/(R-r))}.$$

Lemma 1. [3]. Let $\alpha\epsilon L_{si}$, $\beta\epsilon L_{si}$ and φ be of an analytic in D_R, $R<+\infty$, characteristic function ϕ of probability law F, satisfying condition (5).

If $\beta^{-1}(c\alpha(x))/x\to 0$ and $\alpha\left(x/\beta^{-1}(c\alpha(x))\right)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0;+\infty)$ then $\rho_{\alpha,\beta}^{(R)}[\varphi]=\omega_{\alpha,\beta}^{(R)}[F]$.

Using Lemma 1 we prove the next theorem.

Theorem 2. Let $R_{F_1}=R_{F_2}=R\in(0,+\infty)$ and (5) holds for $F=F_j$, $j=1;2$. Suppose that the functions $\alpha\epsilon L_{si}$ and $\beta\epsilon L_{si}$ satisfy the conditions of Lemma 2 and $\alpha\left(x\alpha^{-1}(c\beta(x))\right)=(1+o(1))c\beta(x)$ as $x\to+\infty$ for each $c\epsilon(0;+\infty)$. Then

$$\omega_{\alpha,\beta}^{(R)}[F_1*F_2]\le\max\{\omega_{\alpha,\beta}^{(R)}[F_1],\omega_{\alpha,\beta}^{(R)}[F_2]\},\qquad(6)$$

and if moreover $\omega_{\alpha,\beta}^{(R)}[F_2]<\omega_{\alpha,\beta}^{(R)}[F_1]$ then

$$\omega_{\alpha,\beta}^{(R)}[F_1*F_2]=\omega_{\alpha,\beta}^{(R)}[F_1]\qquad(7)$$

Proof. Suppose that $\max\{\rho_{\alpha,\beta}^{(R)}[\varphi_1],\rho_{\alpha,\beta}^{(R)}[\varphi_2]\}=\rho<+\infty$. Then $\ln M(r,\phi_j)\le\alpha^{-1}\left((\rho+\varepsilon)\beta(1/(R-r))\right)$ for every $\varepsilon>0$ and all $r\in(r_0(\varepsilon),R)$ and, thus,

$$\ln M(r,\varphi)\le\ln M(r,\varphi_1)+$$

$$+\ln M(r,\varphi_2)\le 2\alpha^{-1}\left((\rho+\varepsilon)\beta(1/(R-r))\right).$$

Since $\alpha\epsilon L_{si}$, hence it follows that $\rho_{\alpha,\beta}^{(R)}[\varphi]\le\rho+\varepsilon$, and in view of the arbitrariness of ε we obtain the inequality $\rho_{\alpha,\beta}^{(R)}[\varphi]\le\rho$, which is obvious when $\rho=+\infty$. Thus, $\rho_{\alpha,\beta}^{(R)}[\varphi]\le\max\{\rho_{\alpha,\beta}^{(R)}[\varphi_1],\rho_{\alpha,\beta}^{(R)}[\varphi_2]\}$, and by Lemma 1 inequality (6) is true.

Further, choosing $r_1=r$ and $r_2=r+(R-r)/2$ from (4) for the function φ_1 we have

$$T(r,\varphi_1)\le\ln^+M(r,\varphi_1)\le\frac{3r+R}{R-r}T\left(r+\frac{R-r}{2},\varphi_1\right)$$

$$\le\frac{4R}{R-r}T\left(r+\frac{R-r}{2},\varphi_1\right),$$

i. e. in view of conditions $\alpha\epsilon L_{si}$ and $\beta\epsilon L_{si}$

$$\rho_{\alpha,\beta}^{(R)}\left[T(r,\varphi_1)\right]:=\overline{\lim_{r\uparrow R}}\frac{\alpha(T(r,\varphi_1))}{\beta(1/R-r)}\le$$

$$\overline{\lim_{r\uparrow R}}\frac{\alpha(\ln M(r,\varphi_1))}{\beta(1/(R-r))}=\rho_{\alpha,\beta}^{(R)}[\varphi_1]\le$$

$$\le\overline{\lim_{r\uparrow R}}\frac{\alpha\left(\frac{2R}{R-r-(R-r)/2}T\left(r+(R-r)/2\right),\varphi_1\right)}{\beta\left(1/\left(R-r-(R-r)/2\right)\right)}\cdot\frac{\beta(2/(R-r))}{\beta(1/(R-r))}$$

$$= \overline{\lim_{r\uparrow R}} \frac{\alpha(T(r,\varphi_1)/(R-r))}{\beta(1/(R-r))}.$$

But by the definition of $\rho_{\alpha,\beta}^{(R)}[T]$ we have $T(r,f) \le \alpha^{-1}(\rho\beta(1/(R-r)))$ for every $\rho > \rho_{\alpha,\beta}^{(R)}[T]$ and all $r \in [r_0(\rho), R)$. Therefore, since $\alpha(x\alpha^{-1}(c\beta(x))) \le (1+0(1))c\beta(x)$ as $x \to +\infty$, we obtain

$$\overline{\lim_{r\uparrow R}} \frac{\alpha(T(r,\varphi_1)/(R-r))}{\beta(1/(R-r))} \le$$

$$\overline{\lim_{r\uparrow R}} \frac{\alpha((1/(R-r)))\alpha^{-1}(\rho\beta(1/(R-r)))}{\beta(1/(R-r))} =$$

$$= \overline{\lim_{x\to+\infty}} \frac{\alpha(x\alpha^{-1}(\rho\beta(x)))}{\beta(x)} = \rho$$

and in view of the arbitrariness of ρ the equality $\rho_{\alpha,\beta}^{(R)}[T] = \rho_{\alpha,\beta}^{(R)}[f]$ is true. Thus, (3) implies the inequality $\rho_{\alpha,\beta}^{(R)}[\varphi_1] \le \max\{\rho_{\alpha,\beta}^{(R)}[\varphi], \rho_{\alpha,\beta}^{(R)}[\varphi_2]\}$, that is by Lemma 1 inequality (6) is proved. If $\omega_{\alpha,\beta}^{(R)}[F_2] < \omega_{\alpha,\beta}^{(R)}[F_1]$ then by this lemma $\omega_{\alpha,\beta}^{(R)}[F_1] = \rho_{\alpha,\beta}^{(R)}[\varphi_1] \le \rho_{\alpha,\beta}^{(R)}[\varphi] = \omega_{\alpha,\beta}^{(R)}[F]$. Theorem 2 is proved.

We remark that for example the functions $\alpha(x) \equiv \ln\ln x$ and $\beta(x) \equiv \ln x$ for $x \ge x_0$ satisfy the conditions of Theorem 2.

2. Connections in terms of convergence classes. Let B be a positive continuously differentiable and increasing to $+\infty$ function on $(0;R)$. If F is a probability law and $R = R_F$ then $\lim\limits_{x\to+\infty} \frac{1}{x}\ln\frac{1}{W_F(x)} = R$. Here we find conditions, under which the correlations

$$\int_{x_0}^{R} \frac{dx}{B\left(\frac{1}{x}\ln\frac{1}{W_{F_j}(x)}\right)} < +\infty, \; j = 1,2 \qquad (8)$$

Imply for $F = F_1 * F_2$ the correlation

$$\int_{x_0}^{R} \frac{dx}{B\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)} < +\infty. \qquad (9)$$

At first we consider a convergence Φ - class. Let $0 < R \le +\infty$ and $\Omega(\mathrm{R})$ be a class of positive unbounded functions Φ on $(0,R)$ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(0;R)$. For $\Phi \in \Omega(\mathrm{R})$, as in [1,4,6], we say that an analytic in D_R function φ belongs to a convergence Φ-class if

$$\int_{r_0}^{R} \frac{\Phi'(r)\ln M(r,\varphi)}{\Phi^2(r)} dr < +\infty \qquad (10)$$

Finally, by $V(R)$ we denote a class of positive continuously differentiable on $(0,+\infty)$ functions v such that $v'(x) \uparrow R$ as $x \uparrow +\infty$. In [4] the following result is proved.

Lemma 2. Let $0 < R \le +\infty$ and the function $\Phi \in \Omega(\mathrm{R})$ satisfies the conditions:

1) the function $\Phi'(r)/\Phi(r)$ is nondecreasing on $[r_0, R)$;

2) $\Phi'(r)(R-r) > 1$ for all $r \in [r_0; R)$;

3) $\Phi'(r + 1/\Phi'(r)) \le H_1\Phi'(r)$ for all $r \in [r_0; R)$, $H_1 = const > 0$;

4) $\frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} \le H_2 < +\infty$ for all $r \in [r_0; R)$;

5) $\int_{r_0}^{R} \frac{\Phi'(r)\ln\Phi'(r)}{\Phi^2(r)} dr < +\infty$

.

Suppose that φ is an analytic in D_R characteristic function on probability law F such that (5) holds. Then in order that ϕ belongs to a convergence Φ-class it is necessary and in the case, when $\ln(1/W_F(x)) = v(x) \in V(R)$ it is sufficient that

$$\int_{x_0}^{R} \frac{dx}{\Phi'\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)} < +\infty. \qquad (11)$$

We remark that the condition 5) in this lemma is unnecessary. Indeed, by condition 4) we have

$$\int_{r_0}^{R} \frac{\Phi'(r)\ln\Phi'(r)}{\Phi^2(r)} dr = \int_{r_0}^{R} \ln\Phi'(r)\, d\left(-\frac{1}{\Phi(r)}\right) =$$

$$= -\frac{\ln\Phi'(r)}{\Phi(r)}\Bigg|_{r_0}^{R} + \int_{r_0}^{R} \frac{d\ln\Phi'(r)}{\Phi(r)} \le$$

$$\le \int_{r_0}^{R} \frac{\Phi''(r)}{\Phi'(r)\Phi(r)} dr + const =$$

$$= \int_{r_0}^{R} \frac{\Phi''(r)\Phi'(r)\Phi(r)}{(\Phi'(r))^2\Phi^2(r)} dr + const \le$$

$$H_2\int_{r_0}^{R} \frac{\Phi'(r)}{\Phi^2(r)} dr + const < +\infty.$$

Theorem 3**.** Let $0 < R \le +\infty$ and the function $\Phi \in \Omega(\mathrm{R})$ satisfy the condition 1)-4) of *Lemma 2*. Let B be a positive continuously differentiable and increasing to $+\infty$ function on $(0,R)$ such that $B(x) \asymp \Phi'(x)$ as $x \to +\infty$. Suppose that $R_{F_j} = R \in (0,+\infty)$,

$\ln\left(1/W_{F_j}(x)\right) = v_j(x) \in V(R)$ and (5) holds for $F = F_j$, $j = 1;2$. Then (8) implies (9).

Proof. Since $B(x) \asymp \Phi'(x)$ as $x \to +\infty$, from (8) for $j = 1;2$ we obtain (11) with $W_{F_j}(x)$ instead $W_F(x)$, and by Lemma 2 for corresponding characteristic function we obtain (10) with φ_j instead ϕ. But $\ln M(r,\varphi) \le \ln M(r,\varphi_1) + \ln M(r,\varphi_2)$. Therefore, (10) holds and by Lemma 2 (11) holds. Since $B(x) \asymp \Phi'(x)$ as $x \to +\infty$, (11) implies (9). Theorem 3 is proved.

Consequence 1. Let $0 < \rho < +\infty$ and F_1 and F_2 be probability laws such that $R_{F_j} = \infty$ and $\ln\left(1/W_{F_j}(x)\right) = v_j(x) \in V(R)$. If $\int\limits_{x_0}^{\infty} W_{F_j}(x)^{\rho/x} dx < +\infty$ then $\int\limits_{x_0}^{\infty} W_{F_1 * F_2}(x)^{\rho/x} dx < +\infty$.

Indeed, if we choose $B(x) = \Phi(x) = e^{\rho x}$ then the function Φ satisfies conditions 1) - 4) of Lemma 2 and $B(x) \asymp \Phi'(x)$ as $x \to +\infty$. Since $B\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right) = \frac{1}{W_F(x)^{\rho/x}}$. Consequence 1 is proved.

We remark that if $R = +\infty$ and $\Phi(x) = e^{\rho x}$ then condition (10) is equivalent to the condition $\int\limits_{r_0}^{\infty} e^{-\rho r} \ln M(r,\varphi)\, dr < +\infty$. A generalization of this correlation is the correlation $\int\limits_{r_0}^{\infty} \left(\alpha(\ln M(r,\varphi))/\beta(r)\right) dr < +\infty$, where $\alpha \in L$ and $\beta \in L$, and if this condition holds then [7]-[9] on definition an entire function ϕ belongs to a generalized convergence $\alpha\beta$ -class. Here we will some modify this definition and will say that an entire function ϕ belongs to a modified generalized convergence $\alpha\beta$ -class if

$$\int\limits_{r_0}^{+\infty} \frac{1}{\beta(r)} \alpha\left(\frac{M(r,\varphi)}{r}\right) dr < +\infty, \ (\alpha \in L, \beta \in L). \qquad (12)$$

The following analog of Lemma 2 is true.

Lemma 3. Let $\alpha \in L^0$ and $\beta \in L^0$ be the continuously differentiable functions, satisfying the conditions: $\alpha'(x) \downarrow \alpha \ge 0$ as $x_0 \le x \to +\infty$, $x\beta'(x)/\beta(x) \ge h > 0$ for $x \ge x_0$ and $\int\limits_{x_0}^{\infty} \left(\alpha(x)/\beta(x)\right) dx < +\infty$. Let φ be an entire characteristic function of probability law F such that $\ln\left(1/W_F(x)\right) = v(x) \in V(+\infty)$ Then in order that φ belongs to the modified generalized convergence $\alpha\beta$ -class it is necessary and sufficient that

$$\int\limits_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right) dx < +\infty, \quad \beta_1(x) = \int\limits_{x}^{\infty} \frac{dr}{\beta(r)} \qquad (13)$$

Proof. In [5, p. 54-55] is proved that $W_F(x)e^{xr} \le 2M(r,\varphi)$ and $M(r,\varphi) \le 1 + W_F(+0) + r\int_0^{+\infty} W_F(x)e^{xr} dx$ for each $r \in [0,+\infty)$ and all $x \ge 0$. We put $\mu(r,\varphi) = \sup\left\{W_F(x)e^{xr} : x \ge 0\right\}$ and $I(r,\varphi) = \int_0^{\infty} W_F(x)e^{xr} dx$. Then

$$\ln\mu(r,\varphi) \le (1+o(1))\ln M(r,\varphi) \le (1+o(1))\ln I(r,\varphi), \quad r \to +\infty. \qquad (14)$$

But

$$I(r,\varphi) = \int\limits_0^{+\infty} W_F(x)\exp\left\{x\left(r+e^{-r}\right)\right\}\exp\left\{-xe^{-r}\right\} dx \le \mu\left(r+e^{-r},\varphi\right)e^r,$$

whence

$$\alpha\left(\left(\ln I(r,\varphi)\right)/r\right) \le \alpha\left(\left(\ln\mu\left(r+e^{-r},\varphi\right)\right)/r + 1\right),$$

and, since $\alpha \in L^0$ and $\beta \in L^0$,

$$\frac{\alpha\left(\left(\ln I(r,\varphi)\right)/r\right)}{\beta(r)} \le (1+o(1))\frac{\alpha\left(\left(\ln\mu\left(r+e^{-r},\varphi\right)\right)/r\right)}{\beta\left(r+e^{-r}\right)}.$$

Hence and from (14) it follows that condition (12) holds if and only if

$$\int\limits_{r_0}^{+\infty} \frac{\alpha\left(\left(\ln\mu(r,\varphi)\right)/r\right)}{\beta(r)} dr < +\infty \qquad (15)$$

As in [4], let $v(r,\varphi)$ be central point of the maximum $\mu(r,\varphi)$ of the integrand. Then [4] $v(r,\varphi) \to +\infty$ as $r \to +\infty$ and $\ln\mu(r,\varphi) = \ln\mu(r_0,\varphi) + \int_{r_0}^{r} \nu(x,\varphi)\, dx$.

Hence

$$\nu(r,\varphi)(r-r_0) \ge \ln\mu(r,\varphi) - \ln\mu(r_0,\varphi) \ge \int_{r/2}^{r} \nu(x,\varphi)\, dx \ge \nu(r/2,\varphi)\, r/2$$

and, since $\alpha \in L^0$ and $\beta \in L^0$, condition (15) holds if and only if

$$\int\limits_{r_0}^{+\infty} \frac{\alpha\left(\nu(r,\varphi)\right)}{\beta(r)} dr < +\infty. \qquad (16)$$

We remark that if $\ln\left(1/W_F(x)\right) = \nu(x) \in V(R)$ then for every $r \in (0,R)$ the function $\ln W_F(x) + rx = -\nu(x) + rx$ has unique point of the maximum $x = \nu(r,\varphi)$ which is increasing and continuous on $(0,R)$, and

$$\ln \mu(r,\varphi) = \max\{\ln W_F(x) + rx : x \ge 0\} = \\ = \ln W_F(\nu(r,\varphi)) + r\nu(r,\varphi) \quad (17)$$

Since $\int_{x_0}^{\infty} dx/\beta(x) < +\infty$, we have $\beta_1(x) = \int_x^{\infty} dr/\beta(r) \downarrow 0$ as $x \to +\infty$ and

$$\int_{r_0}^{+\infty} \frac{\alpha(\nu(r,\varphi))}{\beta(r)} dr = -\int_{r_0}^{\infty} \alpha(\nu(r,\varphi)) d\beta_1(r) = \\ = -\alpha(\nu(r,\varphi))\beta_1(r)\Big|_{r_0}^{\infty} + \int_{r_0}^{\infty} \beta_1(r)\alpha'(\nu(r,\varphi)) d\nu(r),$$

and, since $\alpha(\nu(r,\phi))\beta_1(r) > 0$, hence it follows that condition (16) is equivalent to the condition

$$\int_{r_0}^{\infty} \alpha'(\nu(r,\varphi))\beta_1(r) d\nu(r) < +\infty. \quad (18)$$

From (17) it follows that $W_F(\nu(r,\varphi)) + r\nu(r,\varphi) \ge 0$ for all r enough large. Therefore, $r \ge \frac{1}{\nu(r,\varphi)} \ln \frac{1}{W_F(\nu(r,\varphi))}$ and, thus,

$$\int_{r_0}^{\infty} \alpha'(\nu(r,\varphi))\beta_1(r) d\nu(r) \le \int_{r_0}^{\infty} \alpha'(\nu(r,\varphi))\beta_1 \left(\frac{1}{\nu(r,\varphi)} \ln \frac{1}{W_F(\nu(r,\varphi))}\right) d\nu(r) < +\infty,$$

provided condition (13) holds. The sufficiency of (13) is proved.

Now we prove its necessity.

Since $x = \nu(r,\varphi)$ is a solution of the equation -v(x)+r=0, we have $r = v'(\nu(r,\varphi))$ and from (18) obtain $\int_{r_0}^{\infty} \alpha'(\nu(r,\varphi))\beta_1(v'(\nu(r,\varphi))) d\nu(r) < +\infty$, i.e.

$$\int_{x_0}^{\infty} \alpha'(x)\beta_1(v'(x)) dx < +\infty \quad (19)$$

From a theorem proved in [9] it follows that if $\alpha(x)$ and $\mu(x)$ are continuous functions on $(0, +\infty)$, $-\infty \le A < \alpha(x) < B \le +\infty$, $\mu(x) \downarrow \mu \ge 0$ as $x \to +\infty$, and for a positive function f on (A, B) the function $f^{1/p}$ is convex on (A, B), then

$$\int_0^y \mu(x) f\left(\frac{1}{x}\int_0^x a(t)dt\right) dx \le \left(\frac{p}{p-1}\right)^p \int_0^y \mu(x) f(\alpha(x)) dx, \quad y \le +\infty. \quad (20)$$

We choose, $\mu(x) = \alpha'(x)$, $\alpha(x) = v'(x)$, $f(x) = \beta_1(x)$ and shown that the function $\beta_1^{1/p}$ is convex for some $p > 1$. Indeed,

$$\left(\beta_1^{1/p}(x)\right)'' = \frac{1}{p}\beta_1^{1/p-2}(x)\left(\beta_1(x)\beta_1''(x) - \frac{p-1}{p}\left(\beta_1'(x)\right)^2\right),$$

$$\beta_1(x)\beta_1''(x) - \frac{p-1}{p}\left(\beta_1'(x)\right)^2 = \\ = \frac{1}{\beta^2(x)}\left(\beta'(x)\int_x^{\infty}\frac{dr}{\beta(r)} - \frac{p-1}{p}\right)$$

and in view of the condition $x\beta'(x)/\beta(x) \ge h > 0$ for $x \ge x_0$

$$\beta'(x)\int_x^{+\infty} dr/\beta(r) \ge \beta'(x)\int_x^{2x} dr/\beta(r) \ge x\beta'(x)/\beta(x) \ge h > 0.$$

Therefore, choosing $p > 1$ such that $h - \frac{p-1}{p} \ge 0$, we get the inequality $\left(\beta_1^{1/p}(x)\right)'' \ge 0$ for $x \ge x_0$, that is the function $\beta_1^{1/p}(x)$ is convex and in view of (20)

$$\int_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\int_{x_0}^{x} v'(t)dt\right) dx \le \left(\frac{p}{p-1}\right)^p \int_{x_0}^{\infty} \alpha'(x)\beta_1(v'(x)) dx < +\infty \quad (21)$$

Since

$$\int_{x_0}^{x} v'(t)dt = \ln\frac{1}{W_F(x)} - \ln\frac{1}{W_F(x_0)} = (1+o(1))\ln\frac{1}{W_F(x)}, \quad x \to +\infty,$$

and by condition $\beta \in L^0$ the relation $\beta_1(x(1+o(1))) = (1+o(1))\beta_1(x)$ as $x \to +\infty$ holds, (21) implies (13). The proof of Lemma 3 is completed.

Theorem 4. Let the functions α and β satisfy the conditions of Lemma 3. Suppose that $R_{F_j} = +\infty$, $\ln(1/W_F(x)) = v(x) \in V(+\infty)$ and $\ln(1/W_{F_j}(x)) = v_j(x) \in V(+\infty)$ for $j = 1;2$.

If

$$\int_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{W_{F_j}(x)}\right) dx < +\infty \quad (22)$$

then

$$\int_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right) dx < +\infty \quad (23)$$

Proof. In view of (22) by Lemma 3 the corresponding characteristic functions φ_j belong to the modified generalized convergence $\alpha\beta$-class. Since $\ln M(r,\varphi) \le \ln M(r,\varphi_1) + \ln M(r,\varphi_2)$ and $\alpha \in L^0$, we have

$$\alpha((\ln M(r,\varphi))/r) \le \\ \le \alpha\left(2\max\{(\ln M(r,\varphi_1))/r, ((\ln M(r,\varphi_2))/r)\}\right) \le \\ \le K\max\{\alpha((\ln M(r,\varphi_1))/r), \alpha((\ln M(r,\varphi_2))/r)\} \le \\ \le K\left(\alpha((\ln M(r,\varphi_1))/r) + \alpha((\ln M(r,\varphi_2))/r)\right) < +\infty,$$

whence it follows that φ belongs to the modified generalized convergence $\alpha\beta$-class and, thus, by Lemma 3 (23) holds. Theorem 4 is proved.

Conclusions. Established connections between the decrease of function $W_{F_1 * F_2}$ and the decrease of functions W_{F_1} and W_{F_2} in terms of generalized orders and convergence classes.

References:

1. Filevych P.V., Sheremeta M.M. (2003) On a convergence class for entire functions. Bull. Soc. Lettres Lodz 53 Ser. Rech. Deform., no. 40, pp. 5–16.
2. Gol'dberg A.A., Ostrovskii I.V. (1970) Value distribution of meromorphic functions. Moskow. Nauka. (in Russian).
3. Kinash O.M., Parolya M.I., Sheremeta M.M. (2012) Growth of characteristic functions of probability laws. Dopovidi NAN of Ukraine, no. 8, pp. 13–17.
4. Kulyavets' L.V., Mulyava O.M. Sheremeta M.M. (2013) On belonging of characteristic functions of probability laws to a convergence class. Bull. Soc. Sci. Lett. Lodz, Ser Rech. Deform., vol. 63, no. 2, pp. 9–22.
5. Linnik Ju.V., Ostrovskii I.V. (1972) Decompositon of random variablesand vektors, Moskow. Nauka. (in Russian).
6. Mulyava O.M., Sheremeta M.M. (2000) On a convergence class for Dirichlet series. Bull. Soc. Lettres Lodz 50 Ser. Rech. Deform., no.30, pp. 64–69.
7. Mulyava O.M. (1999) Convergence classes in the thery of Dirichlet series. Dopovidi NAN of Ukraine, no. 3, pp. 35-39.
8. Mulyava O.M. (1999) On convergence classes of Dirichlet series. Ukr. Math. Journ., vol. 51, no. 11, pp. 1485–1494.
9. Mulyava O.M. (2006) Integral analog of one generalization of the Hardy inequality and its applications. Ukr. Math. Journ., vol. 58, no. 9, pp. 1271–1275.
10. Sheremeta M.M. (2003) On two classes of positive functions and belonging to them of main characteristic of entire functions. Mat. Stud., vol. 19, no. 1, pp. 73–82.

Мулява О.М., Шеремета М.М. КОМПОЗИЦІЯ ЙМОВІРНІСНИХ ЗАКОНІВ

Невід'ємна і неспадна, неперервна зліва на проміжку $(-\infty,+\infty)$ функція F називається ймовірнісним законом, якщо $\lim\limits_{x\to+\infty} F(x)=1$ і $\lim\limits_{x\to-\infty} F(x)=0$, а композиція двох ймовірнісних законів F_1 and F_2 визначається рівністю $F(s)=(F_1*F_2)(s):=\int_{-\infty}^{\infty} F_1(x-s)dF_2(x)$. Якщо для $x\geq 0$ ми покладемо $W_F(x)=1-F(x)+ +F(-x)$, тоді $W_F(x)\downarrow 0$ при $x\to+\infty$. У статті досліджено зв'язок між спаданням функції $W_{F_1*F_2}$. і спаданням функцій $W_{F_1}(x)$ and $W_{F_2}(x)$ в термінах узагальнених порядків та класів збіжності. Для цього через L позначимо клас неперервних невід'ємних на $(-\infty,+\infty)$ функцій α таких, що $\alpha(x)=\alpha(x_0)\geq 0$ для $x\leq x_0$ і $\alpha(x)\uparrow+\infty$ при $x_0\leq x\to+\infty$. Кажуть, що $\alpha\in L^0$, якщо $\alpha\in L$ і $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ при $x\to+\infty$. Нарешті, $\alpha\in L_{si}$, якщо $\alpha\in L$ і $\alpha(cx)=(1+o(1))\alpha(x)$ при $x\to+\infty$ для будь-якого фіксованого $c\in(0,+\infty)$, тобто α є повільно зростаюча функція. Поклавши $R_F=\varliminf\limits_{x\to+\infty}((1/x)\ln(1/W_f(x))$, два випадки $R_F=+\infty$ і $R_F<+\infty$ розглядаються окремо. Для $R_F=+\infty$ введено таку характеристику $\omega_{\alpha,\beta}[F]:=\varlimsup\limits_{x\to+\infty}\alpha(x)/\beta((1/x)\cdot\ln(1/W_F(x)))$ і доведено, що якщо $\alpha\in L_{si}$ і $\beta\in L^0$, то $\omega_{\alpha,\beta}[F_1*F_2]\leq\max\{\omega_{\alpha,\beta}[F_1],\omega_{\alpha,\beta}[F_2]\}$ і, крім того, якщо $\omega_{\alpha,\beta}[F_2]<\omega_{\alpha,\beta}[F_1]$, тоді $\omega_{\alpha,\beta}[F_1*F_2]=\omega_{\alpha,\beta}[F_1]$. Якщо $0<R_F=R<+\infty$ і $\varlimsup\limits_{x\to+\infty} W_F(x)e^{Rx}=+\infty$, ми покладемо $\omega^{(R)}_{\alpha,\beta}[F]=\varlimsup\limits_{x\to+\infty}\alpha(x)/\beta(x/\ln^+(W_F(x)\cdot e^{R\cdot x}))$. Доведено, що якщо $R_{F_1}=R_{F_2}=R\in(0,+\infty)$, $\alpha\in L_{si}$, $\beta\in L_{si}$, $\alpha(x\alpha^{-1}(c\beta(x))=(1+o(1))c\beta(x))$ і $\alpha(x/\beta^{-1}(c\alpha(x))=(1+o(1))\alpha(x)$ при $x\to+\infty$ для будь-якого $c\in(0;+\infty)$, тоді $\omega^{(R)}_{\alpha,\beta}[F_1*F_2]\leq\max\{\omega^{(R)}_{\alpha,\beta}[F_1],\omega^{(R)}_{\alpha,\beta}[F_2]\}$ і, крім того, якщо $\omega^{(R)}_{\alpha,\beta}[F_2]<\omega^{(R)}_{\alpha,\beta}[F_1]$, тоді $\omega^{(R)}_{\alpha,\beta}[F_1*F_2]=\omega^{(R)}_{\alpha,\beta}[F_1]$.

Зв'язок між спаданням функції $W_{F_1*F_1}(x)$ і спаданням функцій $W_{F_1}(x)$ і $W_{F_2}(x)$ вивчено також у термінах класів збіжності. За певних умов на функції α, β і $W_{F_j}(x)$ доведено, наприклад, що якщо $R_F=+\infty$ і $\int_{x_0}^{\infty}\alpha'(x)\beta_1((1/x)\cdot\ln(1/W_{F_j}(x)))dx<+\infty$ для $j=1,2$, де $\beta_1(x)=\int_x^{\infty}dr/\beta(r)$, то $\int_{x_0}^{\infty}\alpha'(x)\beta_1((1/x)\cdot\ln(1/W_{F_1*F_2}(x)))dx<+\infty$.

Ключові слова: *ймовірнісний закон, композиція ймовірнісних законів, узагальнені порядки, класи збіжності, спадання функції.*